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**LETTER TO THE EDITOR**

# On the conformal covariant energy–momentum tensor

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**Abstract.** In a conformal symmetric theory we propose a conformal covariant energy–momentum tensor, in terms of which other conformal currents are expressed.

When Noether's theorem is applied to a Lagrangian from which the field equation is covariant under the conformal transformations, it leads to canonical currents such as energy-momentum tensor $T^{c}_{\mu\nu}$, angular momentum tensor $J^{c}_{\mu\nu\lambda}$, dilatation tensor $D^{c}_{\mu}$, and special conformal tensor $K^{c}_{\mu}$ (Mack and Salam 1969). The special conformal tensor, however, is not formally conserved (Ferrara *et al* 1973). In order to get rid of this defect, one can re-define the canonical currents in a symmetric form by adding a complete divergence term which does not change the corresponding global conserved quantites obtained by integration over three-dimensional space (Wess 1960, Gross and Wess 1970). Although this symmetrisation technique for the canonical currents is acceptable, we feel this method is somewhat artificial. We shall here approach this problem in a different way. We propose a conformal covariant energy–momentum tensor, and show how the difficulty with the use of Noether's theorem can be avoided.

The conformal group is a 15-parameter group which includes the following transformations:

(i) Inhomogeneous Lorentz transformations

$$x^{\nu}_{\mu} = \Lambda^{\nu}_{\mu} x_{\nu} + a_{\mu} \tag{1}$$

(ii) dilatation transformations

$$x'_{\mu} = \rho x_{\mu} \tag{2}$$

(iii) special conformal transformations

$$x'_{\mu} = (x_{\mu} + c_{\mu} x^2)/\Omega \qquad \Omega = 1 + 2cx + c^2 x^2 \tag{3}$$

where

$$cx = c_{\mu} x^{\mu} = g^{\mu\nu} c_{\mu} x_{\nu} \qquad g^{\mu\nu} = (+++, -).$$

These transformations satisfy the identity

$$\frac{\partial x'_{\mu}}{\partial x^{\lambda}} \frac{\partial x'_{\nu}}{\partial x^{\sigma}} g^{\mu\nu} = |\det(\partial x'/\partial x)|^{1/2} g^{\lambda\sigma} \tag{4}$$

so that the matrix

$$\Lambda_{\mu\nu}(x) = |\det(\partial x'/\partial x)|^{-1/4}\, \partial x'_{\mu}/\partial x^{\nu} \tag{5}$$

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is a Lorentz matrix. If a set of fields $\phi_\alpha(x)$ belongs to a linear representation of the Lorentz group

$$\phi'_\alpha(x') = D_\alpha^\beta(\Lambda)\phi_\beta(x) \tag{6}$$

then the representation of the conformal group is provided by (Isham *et al* 1970)

$$\phi'_\alpha(x') = |\det(\partial x'/\partial x)|^{\frac{1}{4}l_\phi} D_\alpha^\beta(\Lambda(x))\phi_\beta(x) \tag{7}$$

where $\Lambda(x)$ is given by equation (5), and $l_\phi$ is the conformal weight of the field

$l_\phi = -1$ for the scalar and vector field

$l_\phi = -\frac{3}{2}$ for the spinor field.

For the special conformal transformal transformations, it gives

$$D_\alpha^\beta(\Lambda(x)) = g_\alpha^\beta + (c^\mu x^\nu - x^\mu c^\nu)I_{\mu\nu\alpha}^\beta \tag{8}$$

where $I_{\mu\gamma\alpha}^\beta$ is defined by the transformation properties of the field $\phi_\alpha(x)$. For scalars, $I_{\mu\nu\alpha}^\beta = 0$; for vectors, $I_{\mu\nu\alpha}^\beta = g_{\mu\alpha}g_\nu^\beta - g_{\nu\alpha}g_\mu^\beta$; and for spinors, $I_{\mu\nu\alpha}^\beta = \frac{1}{4}[\gamma_\mu, \gamma_\nu]_\alpha^\beta$.

By using Noether's theorem, we can obtain the canonical currents

$$T_{\mu\nu}^c = g_{\mu\nu}\mathscr{L} - \pi_\mu^\alpha \partial_\nu \phi_\alpha \tag{9}$$

$$J_{\mu\nu\lambda}^c = x_\lambda T_{\mu\nu}^c - x_\nu T_{\mu\lambda}^c + \pi_\mu^\alpha J_{\nu\lambda\alpha}^\beta \phi_\beta \tag{10}$$

$$D_\mu^c = x^\nu T_{\mu\nu}^c \pi_\mu^\alpha l_\phi \phi_\alpha \tag{11}$$

$$K_{\mu\nu}^c = x^2 T_{\mu\nu}^c - 2x_\nu x^\lambda T_{\mu\lambda}^c + \pi_\mu^\alpha(2l_\phi x_\nu \phi_\alpha - 2x^\lambda I_{\nu\lambda\alpha}^\beta \phi_\beta) \tag{12}$$

$$\pi_\mu^\alpha = \partial\mathscr{L}/\partial\partial^\mu\phi_\alpha.$$

The canonical energy–momentum tensor $T_{\mu\nu}^c$ from which the other canonical currents are composed is not conformal covariant. This is unnatural in a conformal symmetric theory. What we want to do now is to construct a conformal covariant energy–momentum tensor $\theta_{\mu\nu}$ by adding to $T_{\mu\nu}^c$ some extra Poincaré covariant terms. Since $\theta_{\mu\nu}$ is Poincaré covariant, and for the dilatation transformations the proof of covariance is trivial, we restrict ourselves to the case of the special conformal transformations.

*The scalar field*

Let us begin with the simplest case of massless scalar field. The Lagrangian is

$$\mathscr{L} = -\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi \tag{13}$$

we then have

$$T_{\mu\nu}^c = -\tfrac{1}{2}g_{\mu\nu}\partial_\lambda\phi\,\partial^\lambda\phi + \partial_\mu\phi\,\partial_\nu\phi. \tag{14}$$

Now we suppose the conformal covariant energy–momentum tensor can be written in the following general form

$$\theta_{\mu\nu} = T_{\mu\nu}^c + a g_{\mu\nu}\partial^\rho(\phi\,\partial_\rho\phi) + b\,\partial_\nu(\phi\,\partial_\mu\phi) \tag{15}$$

where the constants are determined in such a way that $\theta_{\mu\nu}$ is a conformal covariant tensor under the special conformal transformations. It is easy to find the unique solution, namely $a = -b = \frac{1}{3}$. Hence we have

$$\theta_{\mu\nu} = T_{\mu\nu}^c + \tfrac{1}{3}\partial^\rho(g_{\mu\nu}\phi\,\partial_\rho\phi - g_{\rho\nu}\phi\,\partial_\mu\phi) \tag{16}$$

and

$$\theta'_{\mu\nu} = \Omega^4 D^{\lambda\sigma}_{\mu\nu}(\Lambda(x))\theta_{\lambda\sigma}(x) \tag{17}$$

where

$$D^{\lambda\sigma}_{\mu\nu}(\Lambda(x)) = g^{\lambda}_{\mu}g^{\sigma}_{\nu} + 2(c_\mu x^\lambda - x_\mu c^\lambda)g^{\sigma}_{\nu} + 2g^{\lambda}_{\mu}(c_\nu x^\sigma - x_\nu c^\sigma). \tag{18}$$

Furthermore, we can verify that $\theta_{\mu\nu}$ has the properties

$$\theta_{\mu\nu} = \theta_{\nu\mu} \qquad \theta^{\mu}_{\mu} = 0 \qquad \partial^\mu \theta_{\mu\nu} = 0. \tag{19}$$

Here we noted that the conformal covariant tensor for scalar field, equation (16), is identical to the improved energy momentum tensor proposed by Callan *et al* (1970).

Let us now redefine the other conformal currents in terms of $\theta_{\mu\nu}$ as follows

$$J_{\mu\nu\lambda} = x_\lambda \theta_{\mu\nu} - x_\nu \theta_{\mu\lambda} \tag{20}$$

$$D_\mu = x^\nu \theta_{\mu\nu} \tag{21}$$

$$K_{\mu\nu} = x^2 \theta_{\mu\nu} - 2x_\nu x^\lambda \theta_{\mu\lambda}. \tag{22}$$

They have now very simple expressions, and all of them are conserved due to equation (19). The conformal currents of equations (16) and (20)–(22) differ from the original canonical currents of equations (9)–(12) only by terms which do not contribute to the charges.

$$J_{\mu\nu\lambda} = J^{c}_{\mu\nu\lambda} + \tfrac{1}{3}\partial^\rho[x_\lambda(g_{\mu\nu}\phi\partial_\rho\phi - g_{\rho\nu}\phi\partial_\mu\phi) + x_\nu(g_{\rho\lambda}\phi\partial_\mu\phi - g_{\mu\lambda}\phi\partial_\rho\phi)$$
$$+ \tfrac{1}{2}(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\nu}g_{\lambda\rho})\phi^2] \tag{23}$$

$$D_\mu = D^{c}_{\mu} + \tfrac{1}{3}\partial^\rho(x_\mu\phi\partial_\rho\phi - x_\rho\phi\partial_\mu\phi) \tag{24}$$

$$K_{\mu\nu} = K^{c}_{\mu\nu} + \tfrac{1}{3}\partial^\rho[x^2(g_{\mu\nu}\phi\partial_\rho\phi - g_{\rho\nu}\phi\partial_\mu\phi) + 2x_\nu(x_\rho\phi\partial_\mu\phi - x_\mu\phi\partial_\rho\phi)$$
$$+ (x_\mu g_{\nu\rho} - x_\rho g_{\nu\mu})\phi^2] + g_{\mu\nu}\phi^2 \tag{25}$$

so that

$$P_\mu = \int d^3x\, \theta_{0\mu} = \int d^3x\, T^{c}_{0\mu} \tag{26}$$

$$J_{\mu\nu} = \int d^3x\, J_{0\mu\nu} = \int d^3x\, J^{c}_{0\mu\nu} \tag{27}$$

$$D = \int d^3x\, D_0 = \int d^3x\, D^{c}_{0} \tag{28}$$

$$K_\mu = \int d^3x\, K_{0\mu} = \int d^3x\, K^{c}_{0\mu}. \tag{29}$$

*The vector field*

The free electromagnetic Lagrangian is

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{30}$$

By the same method as in the scalar field case, the conformal convariant energy momentum tensor of the electromagnetic field can be found to be

$$\theta_{\mu\nu} = T^{c}_{\mu\nu} + \partial^\rho(F_{\rho\mu}A_\nu) \tag{31}$$

$$T^{c}_{\mu\nu} = -\tfrac{1}{4}g_{\mu\nu}F_{\rho\lambda}F^{\rho\lambda} - F_{\rho\mu}\partial_\nu A^\rho. \tag{32}$$

Here $\theta_{\mu\nu}$ is not only conformal covariant but also gauge independent (Eriksen and Leinass 1980). From the definition of equations (20)–(22), we have

$$J_{\mu\nu\lambda} = J^{c}_{\mu\nu\lambda} + \partial^{\rho}(x_{\lambda}F_{\rho\mu}A_{\nu} - x_{\nu}F_{\rho\mu}A_{\lambda}) \tag{33}$$

$$D_{\mu} = D^{c}_{\mu} + \partial^{\rho}(x^{\lambda}F_{\rho\mu}A_{\lambda}) \tag{34}$$

$$K_{\mu\nu} = K^{c}_{\mu\nu} + \partial^{\rho}(x^{2}F_{\rho\mu}A_{\nu} - 2x_{\nu}x^{\lambda}F_{\rho\mu}A_{\lambda}). \tag{35}$$

*The spinor field*

For the massless Dirac field, the Lagrangian can be written as

$$\mathscr{L} = -\tfrac{1}{2}\bar{\psi}\gamma_{\mu}\partial^{\mu}\psi + \tfrac{1}{2}\bar{\psi}\overleftarrow{\partial}^{\mu}\gamma_{\mu}\psi \tag{36}$$

the conformal energy-momentum tensor has the form

$$\theta_{\mu\nu} = T^{c}_{\mu\nu} + \partial^{\rho}B_{\rho\mu\nu} \tag{37}$$

where

$$T^{c}_{\mu\nu} = \tfrac{1}{2}(\bar{\psi}\gamma_{\mu}\partial_{\nu}\psi - \bar{\psi}\overleftarrow{\partial}_{\nu}\gamma_{\mu}\psi) \tag{38}$$

and

$$B_{\rho\mu\nu} = \tfrac{1}{8}(\bar{\psi}\gamma_{\mu}\gamma_{\nu}\gamma_{\rho}\psi - \bar{\psi}\gamma_{\rho}\gamma_{\nu}\gamma_{\mu}\psi). \tag{39}$$

The currents associated with the Lorentz rotations, dilatations, and special conformal transformations are expressed respectively as

$$J_{\mu\nu\lambda} = J^{c}_{\mu\nu\lambda} + \partial^{\rho}(x_{\lambda}B_{\rho\mu\nu} - x_{\nu}B_{\rho\mu\lambda}) \tag{40}$$

$$D_{\mu} = D^{c}_{\mu} + \partial^{\rho}(x^{\lambda}B_{\rho\mu\lambda}) \tag{41}$$

$$K_{\mu\nu} = K^{c}_{\mu\nu} + \partial^{\rho}[(g^{\lambda}_{\mu} - 2x\nu x^{\lambda})B_{\rho\mu\lambda}]. \tag{42}$$

Equations (19) and (26)–(29) hold both for the vector and spinor fields, and all the conformal currents are conserved.

In a conformal symmetric theory, it seems natural to introduce the conformal covariant energy–momentum tensor, in terms of which the other conformal currents of equations (20)–(22) can be expressed, so that they are all conserved formally. Both the conformal currents and the canonical currents have the same charges. The conformal currents associated with dilatation transformations and special conformal transformations have particularly simple expressions compared with the original canonical forms of equations (11) and (12). The conservation of these currents is directly related to the tracelessness of the conformal energy–momentum tensor.

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